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RESTATEMENT OF THE THEORY OF CULTURAL RULES[1]

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[1] This paper originated from the first author reading M. Dalla Chiara, R, Guintini and R. Greechie, (2004) *Reasoning in Quantum Theory, Sharp and Unsharp Quantum Logics*, Kluwer Academic Publishers, Dordrecht. In particular in personal correspondence Dick Greechie first suggested to look at the notions of GDP, BCK-algebra and MV-algebra which underlie the present approach. . If any parts of this paper are intelligible by others, the author must thank his generous suggestions throughout for making that possible. All errors and assertions made are those of the author.

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*Abstract: We examine a theory of cultural rules as mathematical transforms. Certain cultural rules may be represented as set functions (called here "transforms") between possible structures (called here "configurations" denoted "C") on generations of an evolutionary sequence. If R is a rule and **R** its transform, the outcome of R acting of a starting configuration C is a set denoted **R**C of possible configurations. The smallest fixed point of the transform **R** of a rule R (called the "minimal structure" of that rule) is the descriptive diagram for illustration of the operation of certain rules traditionally used by ethnographers. A combinatorial density computing certain key population statistics of a cultural system is derivable from the minimal structure of the rule, enabling empirically testable (and successfully tested) predictions of observable population measures on systems using that rule. Therefore we may conclude that cultural structure and the uncertainty inherent in cultural systems are but two parts of one framework. Cultural theory thus has a structure in some ways like that of quantum theory, and is a physically testable physical theory. But quantum theory has been under development for a century. The task for a comparable cultural theory is simply to get started.*

RESTATEMENT OF THE THEORY OF CULTURAL RULES

CONTENTS:

0 Motivation

In 1903 Emile Durkheim[2] observed that "As soon as it was established that every people has its own birth-rate, marriage-rate, crime rate, etc., which can be computed numerically and which remain constant so long as the circumstances are unchanged, but which vary from one people to another, it became apparent that these different categories of acts … do not depend only on individual capriciousness but express permanent and well defined social states … ." It has also been known since at least the work of Barbara Ruheman (1945, 1967)[3] and Andre Weil (1949)[4] that certain cultural systems have properties that lend themselves to operator-theoretical mathematical representations. Several authors have developed that approach using mathematical groups and non-associative algebras for cultural description and inference about properties of

[2] Emile Durkheim, "Sociology and the Social Sciences", originally published 1903, republished in *The Rules of the Sociological Method, and Selected Texts on Sociology and its Method*, The Free Press, New York, 1982, page 202.

[3] B. Ruheman 1945 "A Method for Analyzing Classificatory Kinship Systems" *Southwestern Journal of Anthropology*, 1:532-576, and B. Ruheman 1967 "Purpose and Mathematics: A problem in the analysis of classificatory kinship systems", *Bijdragen* 123:83-124.

[4] In the Appendix to C. Levi-Strauss (1949) *Elementary Structures of Kinship* English edition of 1969, Beacon Press, Boston.

kinship systems[5], but the full power of transforms for construction of a theory of culture have never been exploited.

This paper begins to develop such a theory. In doing so, we also show that the two phenomena of cultural structure, and of characteristic social statistics, noted by Durkheim, are part of the same phenomena; indeed, that certain statistics are derivable from the structure. Examples of structure and statistics used here are of marriage rules, and their associated "demographies", since the mathematical theory of such rules is more developed. It is our belief, though yet to be demonstrated, that this approach is not limited to only the class of marriage rules. But also, most human cultures, and many living systems, have in them some method of assigning descent, sibships and marriage (or mating) rules. Thus, a larger theory will necessarily also incorporate the present theory, or something very like it. Because we concentrate on new development, to prove that a theory is possible, we only briefly mention some closely related topics that have been previously developed, including kinship algebra, work on the mathematics of cultural clarity[6], and some of the present author's previous work. These and other topics may be discussed in future papers.

The present paper represents a stage in a progressive development. Initially, we examined whether cultural rules require structures of at least some sufficient size. Those minimal structures are also objects commonly used by ethnographers to illustrate terminologies of kinship systems and operation of marriage rules, and are intended to represent action of rules (or kinship terminologies), not necessarily empirical social networks. However, an initial and simple

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[5] The literature in kinship algebra is very large. One work more often cited below is Gould (2000) *A New System for the Formal Analysis of Kinship*, University Press of American, Lanham, edited and annotated by David Kronenfeld. A very small sampling of the remaining literature includes: Atkins, John and Woodrow Denham, 1981, "Comment" on "Genealogical Structures and Consanguineous Marriage Systems", in *Current Anthropology*, 22(4): 390-391; Courrege, Philippe 1965, "Un Modele mathematique des structures elementairs de parente", in *L Homme* Volume 5 No 3 – 4, pages 248 – 290, (1965), translated to English by D. Read as pages 289 – 338 in Ballonoff, P. A. (ed.) *Genetics and Social Structure*, Dowden, Hutchinson and Ross, Stroudsburgh Pennsylvania (1974).; Lorrain, F. 1969, unpublished manuscript, Harvard University Department of Social Relations; Liu, Pin-Hsiung 1968, "Formal Analysis of Prescriptive Marriage System: The Murngin Case", in *VIII Congr. Anthropolo. Eth. Sci.*, Vol II Ethnology, pages 90 – 92.; Ottenheimer, Martin, with Richard Greechie 1975, "An Introduction to a Mathematical Approach to the Study of Kinship". In P. Ballonoff (ed.), *Genealogical Mathematics*. Paris: Mouton. 1975. Ottenheimer, Martin 1992, *Modeling Systems of Kinship 4.0* (Computer Program in QuickBASIC with documentation). Dubuque: Wm. C. Brown Publishers ; White, Harrison 1996 "Models of Kinship Systems with Prescribed Marriage", in Lazarsfeld, O. F. and N. W. Hendry (eds.) *Readings in Mathematical and Social Sciences*, Prentice Hall, New York. Dwight Read, 2000, "Formal Analysis of Kinship Terminologies and its Relationship to What Constitutes Kinship" in *Mathematical Anthropology and Cultural Theory*, Vol. 1 No. 1 Nov. 2000, http://www.mathematicalanthropology.org.

[6] Ezhkova, Irina, 2002 "Challenges of Cultural Theory: Theory of Cognitive States" pages 423 – 437 in Trappl, R. (ed) *Cybernetrics and Systems 2002*, Vol. 1, Austrian Society for Cybernetic Studies, Vienna. Ezhkova, Irina, 2004, "The Principles of Cognitive Relativity, Rationality and Clarity: Application to Cultural Theory", *Cybernetics and Systems: An International Journal*, Vol. 35 No. 2-3, March-May 2004, pp. 229 – 258, and Ezhkova, Irina 2005, "Self-Organizing Representations" in *Cybernetics and Systems: An International Journal*, Vol. 36 No. 8, pp. 861-875.

empirical prediction was this: if an anthropologist claims that a culture uses some rule, it is an empirically testable proposition if at least the minimally sufficient size required by that rule is present. But while systems usually do not operate at minimal sizes, a much more powerful tool applicable to predicting key demographic observations for populations of any size, also results from the theory, due to a well established if otherwise little used theorem of combinatorics. Empirical tests have shown this prediction is successful.[7] The mathematical reasons this works are cited at Parts 5 and 10 of this paper. Those insights led eventually to the current paper, which restates the foundations.

The restatement is this: every descent sequence following some specific marriage rule(s) can be observed to have on it a mapping called "descent". A "pure" statement of that rule may be represented by a configuration which repeats itself in one generation. By "repeats itself" we mean, the configuration is a fixed point of the transform representing action of the rule. Therefore describing a culture as having a particular marriage rule is equivalent to claiming to observe a particular fixed point of that descent transform, in describing the operation of the rule. Based on the characteristic size of the minimal fixed point, one may in turn compute and predict certain demographic measures. A reader with knowledge of quantum mechanics or quantum logic will readily see that though the above is not quantum mechanics, it has an organization in some ways very much like quantum theory. But quantum theory has had nearly a century to develop, from a previous foundation of more than a century of development of statistical mechanics. Cultural theory is not (yet) as neatly organized as the sometimes similar structures in physics, but the century is new.

1 Notational Conventions

Recall that Z is a <u>relation</u> on a set S when Z is a subset of SxS. For $b,c \in S$, write bZc when (b,c) is a member of Z. The relation Z is <u>symmetric</u> if bZc implies cZb. Z is <u>totally non-symmetric</u> if bZc implies not cZb. Z is <u>reflexive</u> if bZb for all $b \in S$, and <u>totally non-reflexive</u> if not bZb for all $b \in S$. We write bZ for $\{c | bZc\}$ and $\mathfrak{Z}$ for $\{bZ \mid b \in S\}$. The relation Z is <u>transitive</u> if aZb and bZc implies aZc. A <u>partition</u> of a set S is a set $\mathcal{P}$ of non-empty subsets P_i of S such that no P_i is empty, $\bigcup_i P_i = S$ and for any $P_i, P_j \in S$ if $P_i \neq P_j$, then $P_i \cap P_j = \emptyset$. Call one of these subsets $P_i \in S$ a <u>cell</u> of S. An <u>equivalence relation</u> is a relation E on a set S which is reflexive, symmetric and transitive on S. Write #S for the number of elements in a set S. Write ":=" to define the symbol on the left of ":=" by the object on the right.

2 Basic Sets and Relations

A foundation paper for this approach is Ballonoff (2008)[8]. The definitions below follow closely from that paper.

[7] See especially citations at footnotes 13, 21 and 25.

[8] Paul Ballonoff, (2008) "MV-algebra for Cultural Rules", in *International Journal for Theoretical Physics*, Volume 47 No. 1, 223-235, special issue for the 2006 Proceedings of the International Quantum Structures Association.

Let $\mathbb{P}_i$ be a non-empty set whose members are called individuals,[9]

Definition 1: An evolutionary structure $\mathbb{S}_i$ is a quintuple ($\mathbb{P}_i$, $\mathbb{R}_i$, D_i, B_i, M_i) where $\mathbb{R}_i$ is a non-empty set of rules, and D_i, B_i, and M_i are binary relations on $\mathbb{P}_i$. If bD_ic and there exists no $d \in \mathbb{P}_i$, $d \neq b,c$ for which bD_id and dD_ic, then c is a parent of b and b is an offspring of c, which we also denote as cP_ib. A rule is a statement describing how to form the D_i, B_i, and/or M_i relations without violation of the four axioms listed next. Using this terminology the axioms are:

1. D_i is totally non-symmetric and transitive;
2. M_i is transitive and symmetric;
3. If $b,c,d \in \mathbb{P}_i$, and both dP_ib and dP_ic then bB_ic;
4. $\#bM_i \leq 2$.

The above notation allows that there may be more than one evolutionary structure, such as $\mathbb{S}_1$, $\mathbb{S}_2$, etc. For purposes of this paper we assume a finite number of evolutionary structures. Since in most cases we are discussing just one evolutionary structure, in most of the below, when context permits to do so without ambiguity, we drop the subscript denoting the particular evolutionary structure.

Since the relation D is totally non-symmetric this implies that D is totally non-reflexive. Note, if bMc and cPd, then bPd. We interpret the relation M as "marriage", a particular set M as "a marriage", the relation B as "sibling of", and a given set B as a "sibship". (named using the middle letter "siBship"). Note that one result of Definition 1.4 is that a marriage is "monogamous" and "permanent" between any pair of "married" individuals. The definition of a "rule" implicitly restricts the present paper to consideration only of rules related to "marriage" and "kinship", but does not otherwise restrict how such rules may be stated.

Note especially that the term "descent" here means "ascription" in a cultural sense, and may not necessarily mean biological descent. A cultural ascription of descent could be a rather simple empirical statement, such as "John is the son of Mary". Of much more interest are ascriptions tied to rules of descent such as: "two individuals can only marry and be ascribed offspring if they are not first cousins or closer relatives" together with an ascription such as, "individuals b, c and d are offspring assigned to a particular such marriage". Many forms of rules of ascription of descent are possible. If what we mean is actual biological descent, then that will be the ascription explicitly used. In many cultural rules, the ascription of descent differs from the biology (such as adoption, as a simple example).[10]

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9 The notation "b" for an individual member of $\mathbb{P}$ is a simplification of the notation $|b\rangle$ or $\langle b|$ used in Ballonoff, P. (1976) *Mathematical Foundations of Social Anthropology*, Mouton Paris.

10 This framework however will also allow later application to biological evolution, in which case we might select pure genetic mating, instead of the ascription used in the present presentation. In his operator formulation of population genetics, Y.I. Lyubich, *Mathematical Structures in Population Genetics*, Springer Verlag, Berlin (1992),

We defined that a rule is any statement in natural, logical or mathematical language which states how a relationship on a given generation may form, which does not violate the axioms in Definition 1. For example, the above statement excluding first cousins as marriage partners is a rule that says how the *M* relation may form. In what follows we especially study rules of how the *M* relation may form, because, as noted earlier, marriage rules are already better known mathematically. By using that example, we do not imply that "marriage rules" are the only possible rules.

Given a non-empty subset $\mathbb{G}^t$ of $\mathbb{P}$, we say that a relation *Z* respects $\mathbb{G}^t$ in case, for all $b \in \mathbb{G}^t$, b*Z* is either a subset of $\mathbb{G}^t$ or disjoint from $\mathbb{G}^t$, but not both. Given a family of subsets $\mathbb{G}^t \in \mathbb{P}$, (indexed by a set $t \in \mathcal{T}$ of consecutive non-negative integers) we say that a pair of relations *Z*,*W* splits each $\mathbb{G}^t$ in case, for all $t \in \mathcal{T}$ and for all $b \in \mathbb{G}^t$, $bZ \cap bW = \varnothing$. If a rule $R \in \mathbb{R}$ governs formation of the relationship *Z* then we say R determines *Z*.

Definition 2: Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $\mathbb{G}^t \subseteq \mathbb{P}$, and let $\mathcal{T}$ be a set of consecutive non-negative integers. For $t \in \mathcal{T}$, let $\mathbb{G}^t$ be a family of non-empty subsets of $\mathbb{P}$. Then $\mathcal{G}=\{\mathbb{G}^t \mid t \in \mathcal{T}\}$ is called a descent sequence of $\mathbb{S}$ in case, for all $\mathbb{G}^t \in \mathcal{G}$,

1. *M* and *B* respect $\mathbb{G}^t$;
2. the pairs *D*,*B* and *D*,*M* split $\mathbb{G}^t$: and
3. when $\mathbb{G}^{t-1}, \mathbb{G}^t \in \mathcal{G}$; $b \in \mathbb{G}^t$; and c*P*b, then $c \in \mathbb{G}^{t-1}$.

If $\mathcal{G}=\{\mathbb{G}^t \mid t \in \mathcal{T}\}$ is a descent sequence and $\mathbb{G}^t \neq \varnothing$, $t \in \mathcal{T}$ then $\mathcal{G}$ contains at least one non-empty set and thus is non-empty. Notice especially, that the set $\mathbb{G}^{t+1}$ thus contains all of, and only, the immediate descendants of individuals in $\mathbb{G}^t$. When needed for clarity, if $\mathbb{S}_i$ is an evolutionary structure we denote the jth descent sequence of $\mathbb{S}_i$ as $\mathcal{G}_{ij}$ and the tth generation of $\mathcal{G}_{ij}$ as $\mathbb{G}_{ij}{}^t$.

The concepts of "split" and "respect" induce a *generational coherence of cells*: The effect of these notions is to require that the cells B each occur in only one generation, that the subsets M each occur in only one generation, and that if $M \subset \mathbb{P}$, $B \subset \mathbb{P}$ and M contains the parents of the individuals in B, then the members of M and of B are not in the same generation.

The set $\mathcal{T}$ is called the local calendar of $\mathcal{G}$. For each evolutionary structure $\mathbb{S}$ we assume a common calendar $\mathcal{T}_s$ of consecutive nonnegative integers $t_s \in \mathcal{T}_s$ such that the smallest has $t_s=0$, which denotes the "oldest" generation of the evolutionary structure. A given $t_s \in \mathcal{T}_s$ is called a

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lays out essentially the same basic working space as used here, including the discrete generation assumption, but with real continuous time. Lyubich assumes panmictic populations, that is, biologically randomly mating populations, whereas here, we do not necessarily discuss the genetic mating structure; just the culturally ascriptive marriage rules. Development of these issues is reserved for a subsequent paper discussing biological evolution.

common calendar date, and a given $t \in \mathcal{T}$ is called a local calendar date. We will start each local calendar with t=0, and set the following relationship between the local dates and the common calendar dates. We assume that each descent structure $\mathcal{G}$ of **S**, with local calendar $\mathcal{T}$, has associated with it a non-negative integer κ such that for each $t \in \mathcal{T}$ in a local calendar we can find a corresponding common calendar date t_s=t+κ, and for which κ=0 only for a descent sequence whose initial generation is the oldest generation of the evolutionary structure, or a subset of it. The integer κ is called the calendar constant of $\mathcal{G}$. The local calendar dates of each descent sequence of **S** accordingly can be adjusted to the common calendar dates.

To specify these facts in our notation, if descent sequence $\mathcal{G}_{ij}$ of evolutionary structure $\mathbf{S}_i$ has calendar constant κ, denote these facts as $_\kappa\mathcal{G}_{ij}$ and denote the t[th] generation of $_\kappa\mathcal{G}_{ij}$ as $_\kappa\mathbb{G}_{ij}^{t}$. Thus the common calendar date of the generation $_\kappa\mathbb{G}_{ij}^{t}$ can be easily identified as t+κ, and the initial date of $_\kappa\mathcal{G}_{ij}$ (the date of the generation denoted as $_\kappa\mathbb{G}_{ij}^{0}$) in the common calendar is κ.

The above assumption that there is a unique "oldest" generation of each evolutionary structure, combined with the assumption that the dates can only be non-negative integers, implies that the "history" of any evolutionary structure is finite in the "backward" direction. We can express this fact by saying that in this paper we are only discussing "forward" descent sequences. While we define an arbitrary start for any descent sequence, discovery of conditions under which an evolutionary structure may continue its existence "forward" are a primary purpose of this research.

Definition 3: Let $\mathbf{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $\mathcal{G}=\{\mathbb{G}^t \mid t \in \mathcal{T}\}$ be a descent sequence of **S** with local calendar $\mathcal{T}$. Then let:

1a. $\mathcal{M}^* := \{bM \mid b \in \mathbb{P}\}$ be the set of all marriages in **S**,

1b. $\mathcal{M} := \{M \mid M \in \mathcal{M}^*, \text{ and for } b \in M\ \exists(d)(d \in \mathbb{P} \text{ and } dDb)\}$ be the set of all reproducing marriages in **S**,

2. $\mathcal{B} := \{bB \mid b \in \mathbb{P}\}$ be the set of all sibships in **S**,

3. $\mathcal{M}^t := \{M \mid M \in \mathcal{M} \text{ and } M \subseteq \mathbb{G}^t\}$ be the set of all reproducing marriages in the t[th] generation of the descent sequence $\mathcal{G}$ of **S**,

4. $\mathcal{B}^t := \{B \mid B \in \mathcal{B} \text{ and } B \subseteq \mathbb{G}^t\}$ be the set of all sibships in the t[th] generation of the descent sequence $\mathcal{G}$ of **S**,

5. $\mathbb{G} := \bigcup_t \mathbb{G}^t\ t \in \mathcal{T}$ is the population of the descent sequence $\mathcal{G}$ of the evolutionary structure **S**.

In this paper we do not further discuss the larger set $\mathcal{M}^*$ of all marriages; we discuss only the reproducing marriages in $\mathcal{M}$ (though of course $\mathcal{M} \subseteq \mathcal{M}^*$). These definitions imply that if **S** is an evolutionary structure with common calendar $\mathcal{T}_S$ then:

$\mathcal{M} = \bigcup \mathcal{M}^t$, $t \in \mathcal{T}$, for T the set of all marriages in **S**, and

$\mathcal{B}=\bigcup \mathcal{B}^t$, $t\in \mathcal{T}$, the set of all sibships in **S**,

and when evolutionary structure $\mathbf{S}_i$ has descent sequences $\mathcal{G}_{ij}$ then $\mathbf{P}_i=\bigcup_j \mathbf{G}_{ij}$. Also as a result of this definition and the concepts "split" and "respect, each descent sequence $\mathcal{G}$ partitions its total population $\mathbf{G}:=\bigcup_t \mathbf{G}^t$.

3 Independent Descent Sequences and Subsequences

We wish to be able to discuss "independent" sequences within an evolutionary structure, and "subsequences" of a given descent sequence. For purposes of this section we adopt the following conventions:

$\mathbf{S}=(\mathbf{P}, \mathbf{R}, D, B, M)$ is an evolutionary structure with common calendar $\mathcal{T}_S$.

$\mathcal{G}=\{\mathbf{G}^t \mid t\in \mathcal{T}_G\}$ is a non-empty descent sequence of **S** with local calendar $\mathcal{T}_G$ and calendar constant η.

$\mathcal{H}=\{\mathbf{H}^t \mid t\in \mathcal{T}_H\}$ is a non-empty descent sequence of **S** with local calendar $\mathcal{T}_H$ and calendar constant κ.

$\mathcal{L}=\{\mathbf{L}^t \mid t\in \mathcal{T}_K\}$ is a non-empty independent descent sequences of **S** with local calendar $\mathcal{T}_L$ respectively, calendar constants λ.

$\mathbf{G}=\bigcup_t \mathbf{G}^t$ $t\in \mathcal{T}_G$ is the population of $\mathcal{G}$.

$\mathbf{H}=\bigcup_t \mathbf{H}^t$ $t\in \mathcal{T}_H$ is the population of $\mathcal{H}$.

$\mathbf{L}=\bigcup_t \mathbf{L}^t$ $t\in \mathcal{T}_L$ is the population of $\mathcal{L}$.

Definition 4:

1. Two descent sequence $\mathcal{G}$ and $\mathcal{H}$ are called <u>independent descent sequences</u> of **S** in case $\mathcal{G}\cap \mathcal{H}=\varnothing$.
2. A descent sequence $\mathcal{H}$ is_a <u>subsequence</u> of a descent sequence $\mathcal{G}$ iff there exists an $\mathbf{H}^t\in \mathcal{H}$, $b\in \mathbf{H}^t$, $b\in \mathbf{G}$ and there exists $c\in \mathbf{G}$ such that bDc.
3. $\mathcal{H}$ and $\mathcal{L}$ are <u>independent subsequences</u> of $\mathcal{G}$ in case $\mathcal{H}$ and $\mathcal{L}$ are independent descent sequences of **S**, $\mathcal{H}\subseteq \mathcal{G}$, $\mathcal{L}\subseteq \mathcal{G}$ and $\mathcal{H}\cap \mathcal{L}=\varnothing$.
4. Descent sequence $\mathcal{H}$ and $\mathcal{L}$ are said to have <u>separated</u> from descent sequence $\mathcal{G}$ at common calendar date λ in case $\mathcal{H}$ and $\mathcal{L}$ are <u>independent subsequences</u> of $\mathcal{G}$ and $\eta=\gamma$, and thus that $\gamma=\eta=\lambda$.

If $\mathcal{G}$ is a descent sequence with local calendar $\mathcal{T}$ and calendar constant κ, we can identify a subset of contiguous generations $\mathcal{K}\subseteq \mathcal{G}$ starting at some generation $k\in \mathcal{T}$ as an independent subsequence $\mathcal{K}$ of $\mathcal{G}$ by selecting the generation $\mathbf{G}^k$ as the initial or 0 generation of the subsequence by setting $\mathbf{G}^k=\mathbf{K}^0\in \mathcal{K}$, assigning $\mathcal{K}$ the calendar constant $\kappa+k$, and applying the normal rules of counting forward from the newly created initial date t=0 to assign remaining values in the new local calendar $\mathcal{T}_K$.[11]

Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $c,d \in \mathbb{P}$. Let $\mathcal{G}_c$ be a descent sequence with calendar constant κ and containing c and let $\mathcal{G}_d$ be a descent sequence with calendar constant λ containing d. Let $\mathcal{C}={}_\kappa\mathcal{G}_c$ and let $\mathcal{D}={}_\lambda\mathcal{G}_d$. If $\mathcal{C} \cap \mathcal{D} \neq \varnothing$ we shall say that $\mathcal{C}$ and $\mathcal{D}$ are of the <u>same line of descent</u>. If there exists a common calendar date such that κ=λ and such that $\mathcal{C} \cap \mathcal{D}=\varnothing$ then we say that $\mathcal{C}$ and $\mathcal{D}$ are <u>different lines of descent</u> at and after date κ. (If the assignment of descent is purely biological we can also refer to $\mathcal{C}$ and $\mathcal{D}$ as <u>species</u>). Note that if $\mathcal{C}$ and $\mathcal{D}$ are different lines of descent of the same evolutionary structure $\mathbb{S}$ after date κ then they are also each independent descent sequences of $\mathbb{S}$ at and after κ. If κ is the smallest such date for which $\mathcal{C}$ and $\mathcal{D}$ are of different lines of descent starting at κ, then the lines of descent <u>separated</u> at common calendar date κ.

Nothing in the above definitions prohibits that a descent sequence "temporarily" separate at some common calendar date κ, remain separated for some finite period k>0, and rejoin at date κ+k. Thus in the period κ to κ+k there are two (or more) descent sequences which are independent descent sequences separated at κ and thus also are subsequences which are only independent sequences for the period κ to κ+k.

4 Origin of Descent Sequences

Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $\mathcal{G}$ be a non-empty descent sequence of $\mathbb{S}$ with calendar constant κ>0, and let $\mathbb{G}^0 \in \mathcal{G}$ be the non-empty initial generation of $\mathcal{G}$. For the present paper we wish to preclude such "spontaneous creations" other than the initial oldest generation. Therefore we assert as follows:

> *Axiom (Darwinian Sequences)*: Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $\mathcal{K}$ be a non-empty descent sequence of $\mathbb{S}$ with calendar constant κ>0. Then there exists a non-empty descent sequence $\mathcal{G}$ of $\mathbb{S}$ such that $\mathcal{G}$ has calendar constant 0, and $\mathcal{K}$ is a subsequence of $\mathcal{G}$.

The Darwinian Sequences axiom says that all descent sequences of a given evolutionary structure can be traced back through a chain of descent in an unbroken series of non-empty generations, to the same common calendar date 0 as the date of initial origin. Note that the Darwinian Sequences axiom does not prohibit that there have been more than one simultaneous "origin of life" in an evolutionary structure, since the axiom does not prohibit that $\mathbb{S}$ contains two (or more) descent sequences each with calendar constant 0 and which mutually separated at

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[11] We knowingly thus also make the implicit assumption, for present purposes, that all descent sequences of a given evolutionary structure have the same "real time" interval T between generations. See also Part 6.

common calendar date 0. However, the Darwinian Sequences axiom states that all such events occur within the same generation, namely, that with common calendar date 0.

Let $\mathbf{S}_i=(\mathbf{P}_i, \mathbf{R}_i, D_i, B_i, M_i)$ be an evolutionary structure with common calendar $\mathcal{T}_s$, and let $\mathcal{G}_{ij}$ be the j[th] descent sequence of $\mathbf{S}_i$. Let $\mathcal{B}_i=\{bB_i \mid b\in\mathbf{P}_i\}$ be the set of all sibships in $\mathbf{S}_i$, and let ${}_1\mathbf{P}_i=\{b \mid b\in\mathbf{G}_{ij}{}^t, \mathbf{G}_{ij}{}^t\in\mathcal{G}_i, t\in\mathcal{T}_s, t>0\}$ be the set of all individuals in $\mathbf{P}_i$ in all generations other than a generation with common calendar date 0. Then $\mathcal{B}_i$ partitions ${}_1\mathbf{P}_i$. But $\mathcal{B}_i$ does not in general partition $\mathbf{P}_i$ since for the 0[th] generation of the common calendar we in general do not have information on formation of sibships. Let $j\in J$ index all descent sequences $\mathcal{G}_{ii}$ of evolutionary structure $\mathbf{S}_i$, and let $t\in\mathcal{T}_{ij}$, where $\mathcal{T}_{ij}$ is the local calendar for $\mathcal{G}_{ii}$ when restated as the common calendar dates of $\mathcal{T}_s$. Let $\mathcal{G}_i{}^t=\bigcup_{j\in J}\mathbf{G}_{ij}{}^t$ be the set of all individuals in $\mathbf{P}_i$ in a generation of common calendar date t, and then let $\mathcal{G}_i{}^*=\{\mathcal{G}_i{}^t, t\in\mathcal{T}_s\}$; then as a result of the Darwinian Sequences axiom and the concepts splits and respects, $\mathcal{G}_i{}^*$ partitions $\mathbf{P}_i$.

5 Descent Map Definition

"Descent" means the ascription of which objects descended from which other objects. We need a convention to describe this. The map defined below does this (see also Ballonoff (2008) per footnote 8, at Definition 6).

> *Convention A*: Let $\mathbf{S}=(\mathbf{P}, \mathbf{R}, D, B, M)$ be an evolutionary structure and let $\mathcal{G}_i$ be a descent sequence of $\mathbf{S}$ having local calendar $\mathcal{T}_i$ and population $\mathbf{G}_i$. .
> Let
> $$\mathbf{D}_i := \mathcal{B}_i \rightarrow \mathcal{M}_i$$
> be the map from the subsets $B\in\mathcal{B}_i$ of sibships of $\mathbf{G}_i$ onto the reproducing subsets $M\in\mathcal{M}_i$ of individuals in $\mathbf{G}_i$ that are ascribed as the parents of the sibships $B\in\mathcal{B}_i$. Call $\mathbf{D}$ the <u>descent map</u> ("descendant of" map) on $\mathbf{G}_i$. Correspondingly between any two adjacent generations $\mathbf{G}_i{}^t,\mathbf{G}_i{}^{t-1}\in\mathcal{G}_i$, let $\mathcal{B}_i{}^t$ be the set of sibships $B\in\mathbf{G}_i{}^t$ that partition $\mathbf{G}_i{}^t$, and let $\mathcal{M}^{t-1}$ be the set of reproducing marriages $M\in\mathbf{G}^{t-1}$. Then $\mathbf{D}_i{}^t:=\mathcal{B}_i{}^t\rightarrow\mathcal{M}_i{}^{t-1}$ is the map associating particular sibships in one generation to their particular sets of parents in the previous generation.

In general since we will be talking about descent maps that act only on a given descent sequence, which descent sequence is specified, we shall not use the subscripts. Note that for $\mathbf{G}_i{}^0$ when t=0 in the common calendar, then $\mathbf{G}_i{}^0$ has no sibships defined on it, but sibships are defined on all other generations.

Since we recognize only reproducing M-sets, the descent map is onto $\mathfrak{M}^{t-1}$, and therefore also, the descent map is a surjection;[12] this fact has strong consequences we note further in Part 10.

Note that the maps $\mathbb{D}$ and $\mathbb{D}^t$ act on sets of subsets B and M of $\mathbb{P}$, while the relations *D* act on individuals $b \in \mathbb{P}$. We shall require that $\mathbb{D}$ and $\mathbb{D}^t$ <u>preserve</u> the relations *D*. That is, $\mathbb{D}(B)=M$ iff $b,c \in \mathbb{P}$, bDc, cPb, $b \in B \in \mathfrak{B}$ and $c \in M \in \mathfrak{M}$. And thus also for given $t \in \mathcal{T}$, $\mathbb{D}^t(B)=M$ iff $b,c \in \mathbb{P}$, $\mathbb{G}^t$, $\mathbb{G}^{t-1} \in \mathfrak{G}$ $b \in \mathbb{G}^t$, $c \in \mathbb{G}^{t-1}$, c is a parent of b (that is, cPb), $b \in B \in \mathfrak{B}^t$ and $c \in M \in \mathfrak{M}^t$.

Therefore $\mathbb{D}$ simply collects all of the relations *D* between each pair of successive generations of a descent sequence, and maps them all simultaneously, as the maps $\mathbb{D}^t$ between particular successive pairs of generations. Since each set B is an equivalence class, each $M \in \mathfrak{M}^{t-1}$ has mapped onto it exactly one $B \in \mathfrak{B}^t$, so also $\mathbb{D}$ is 1-1. As well, we can create the inverse map $\mathbb{D}^{-1}: \mathfrak{M}^{t-1} \rightarrow \mathfrak{B}^t$ called the <u>ancestor</u> map ("ancestor of" map) on $\mathbb{P}$, which we also require to preserve *D*, and thus which is also 1-1 and onto $\mathfrak{B}^t$. Therefore also all of $\mathbb{D}$, $\mathbb{D}^{-1}$ and their specific forms $\mathbb{D}^t$ and $(\mathbb{D}^t)^{-1}$, are bijections.

In what follows we shall often discuss a particular line of descent that is implicitly treated as having separated at some previous date. For example, we may consider a particular descent sequence of a population following a particular cultural rule or set of cultural rules, from the date it starts using that rule or set of rules. In most cases we discuss anthropomorphically as if the species of interest is humans, and most of the examples are of cultural rules found or describable among humans. But in general those are anthropomorphisms. Nothing restricts the descent lines or species to be only humans. Some results are already known in the theory of cultural rules that discuss two very different biological species within the same evolutionary structure.[13] In the mathematical theory of evolution (population genetics) and of biological evolution generally, the process of branching into separate chains of species are examples of this condition.

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[12] Because the descent map is a surjection, and also because $\mathfrak{B}^t$ partitions a generation $\mathbb{G}^t$, the density function of numbers of ways of filling B-sets with individuals of a generation in a descent sequence of an evolutionary structure, is given by the Stirling Number of the Second Kind. See for example van Lint, J. H. and R.M Wilson 1992 *A Course in Combinatorics*, Cambridge University Press, page 106) or Grimaldi, R. P., 1989 *Discrete and Combinatorial Mathematics*, Addison Wesley, Reading Ma., page 178 and Peter Hildon, Jean Peterson and Jurgen Stiger,"On Partitions, Surjections and Stirling Numbers", in *Bulletin of the Belgian Mathematical Society* Vol 1, 1994, 713-735.

[13] Ballonoff, P. 2000 "On The Evolution of Self-Awareness" pages 347 – 352 in Trappl (ed.) *Cybernetics and Systems 2000*, Austrian Society for Cybernetic Research, proceedings of the European meetings on Cybernetics and Systems Research 2000, applies the theory to evolution of social insects and of humans, both of the same evolutionary structure- namely the origin of life on Earth - but of different species and descent lines arising from that.

6 Simple Numerical Properties of Descent Sequences

We shall occasionally need to do some accounting on these sets. Thus:

Definition 5: Let $\mathbb{S}$=($\mathbb{P}$, $\mathbb{R}$, *D*, *B*, *M*) be an evolutionary structure, and let $\mathcal{G}$ be a descent sequence of $\mathbb{S}$. Let $\mathcal{M}^t$ and $\mathcal{B}^t$ be sets defined on generations in $\mathcal{G}$ as in Definition 3. Then for $\mathbb{G}^t \in \mathcal{G}$ let $\gamma^t := \#\mathbb{G}^t$, let $\beta^t := \#\mathcal{B}^t$, and let $\mu^t := \#\mathcal{M}^t$.

Let $\mathbb{S}$ be an evolutionary structure, let $\mathcal{G}$ be a descent sequence of $\mathbb{S}$, let $\mathbb{G}^t \in \mathcal{G}$ be a generation of $\mathcal{G}$. Then $\mu^{t-1} \geq \beta^t$. Since we assume all individuals are descended from at least one parent, therefore, for the set of sibships $\mathcal{B}^t$ of $\mathbb{G}^t$, there must be at least as many sets of parents possible in $\mathbb{G}^{t-1}$ as there are B sets in $\mathbb{G}^t$. The proper relationship in the general case is $\mu^{t-1} \geq \beta^t$ if we admit the possibility that there are some marriages in $\mathbb{G}^{t-1}$ that have no offspring. But when restricted, as we do, only to reproducing M sets, then the equality holds.

The discrete generation discussion above does not deny the existence of "real" time. The "real" time interval T between t and t+1 is called the generation interval. We are thus always capable of converting any discussion into a representation in which the time indices from date t to date t+k are replaced by some appropriate multiple such as kT years (if T is measured in years), where T may be any positive real number and k may be any non-negative integer. Except that we occasionally refer to the real-time generation interval T, we shall not use the real or continuous time version of generational intervals in the present paper.

7 Configurations

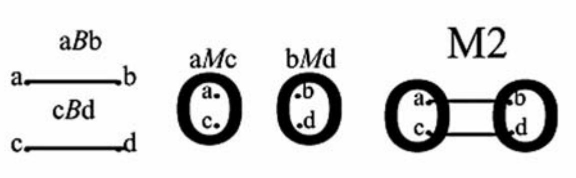
Part 7 defines "configurations" which describe the relations *M* and *B* found on generations or subsets of generations and discusses arithmetic operations on them. Part 7.1 describes a certain class of configurations by use of ordered lists we call "vectors". Part 7.2 defines addition on these vectors.

7.1 Vectors of Configurations

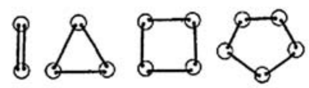
Let $\mathbb{S}$=($\mathbb{P}$, $\mathbb{R}$, *D*, *B*, *M*) be an evolutionary structure and let $\mathcal{G}$ be a descent sequence of $\mathbb{S}$ having local calendar $\mathcal{T}$ and calendar constant κ. Then each $\mathbb{G}^t \in \mathcal{G}$ for $t+\kappa>0$ has on it a partition $\mathcal{B}^t$. $\mathbb{G}^t$ also may also contain some sets of pairs $M \in \mathcal{M}^t$ which are parents of sibships of the following generation, but are not parents of any sibships in $\mathbb{G}^t$. When $t+\kappa>0$, then a concrete configuration $C_t:=(\mathcal{B}^t,\mathcal{M}^t)$, is the pair consisting of the partition $\mathcal{B}^t$ and the sets $M \in \mathcal{M}^t$ on $\mathbb{G}^t$.[14]

[14] The literature on kinship algebra then uses such relations to define objects, such as mathematical groups describing marriage rules. See for example G. DeMeur and A. Gottscheiner (2000) "Prescriptive Kinship Systems, Permutations, Groups and Graphs" in *Mathematical Anthropology and Cultural Theory*, Vol.1, No. 1 November 2000. http://www.mathematicalanthropology.org

An important subset of concrete configurations are those constructed only from the closed (cyclic) objects which we shall call regular structures, and which are similar to configurations commonly used for descriptive purposes in ethnographies.[15] Use a dot to represent an individual, a circle around two dots (say, b and c) to show that b*M*c, and use a line between two dots (say, d,e) to show that they are of sibs in the same sibship d*B*e. When these form simple closed cycles (that is a set of relations such as {b*B*c, c*M*d, d*B*e, e*M*b} closing "back" to the first listed sibship) we give them names M1, M2, M3, etc. and in general Mn, where n= the number of M sets in each closed figure. Thus if a*B*b, c**B**d, a*M*c and d*M*b, the diagram of the relations and the resulting M2 regular structure is:



Thus



M2 M3 M4 M5

are the regular structures with 2, 3, 4 and 5 M-sets, represented by the circles In such diagrams, each straight line represents a distinct sibship, each circle represents a distinct marriage-set. Each regular structure has the same number of B-sets as M-sets, but for naming purposes we only count the M-sets. Note that while each regular structure type has a name, within each the "edges" and "corners" are unlabeled and thus each configuration is an unlabeled diagram. All individuals in a particular regular structure are necessarily of the same generation (as a consequence of the concepts "respect" and "split"). A particular generation might be comprised of none, one or more than one of any particular regular structure, and may contain more than one kind of regular structure. Also, notice that while the rules might, and in general do, specify "sex" of marriage partners, the resulting pictures simply illustrate the existence of the relation;

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[15] Ethnographers commonly represent individuals with small circles or triangles, connect such circles or triangles with a horizontal line to depict a sibship, use a "=" symbol between a circle and a triangle to represent a marriage, and arrange such diagrams in horizontal layers connected by vertical or angled lines connecting a "=" symbol (a particular marriage) to the horizontal line connecting members of a sibship assigned as containing the descendants of that marriage. Such angled or vertical lines thus represent our descent relations *D*, the horizontal lines our relations *B*, the "=" symbols our relations *M*, each layer depicts a generation, and the entire layered sequence depicts a descent sequence or a part of one. Apart from choice of graphic objects, the principal difference in our pictures from those of traditional ethnography is that the traditional pictures look on the descent sequence in a "vertical" position as seen from its "edge", thus exposing the generations as "layers", while ours looks on each generation from "above" thus exposing its internal structure. This however allows us to concentrate on the configurations of particular generations, rather than the viewpoint typical of ethnography, which is instead most often used to study labels (kinship terminologies) and their relationship to marriage rules, either or both of which may be described on the diagram.

they do not per se indicate “sex”.[16] We refer to regular structures collectively as configurational elements.

Henceforth, we shall consider only configurations consisting of regular structures. This may seem like, and indeed is, a severe restriction if our purpose were to describe empirical networks of a particular culture, the common task of ethnography. But we shall see that this none the less allows us to get quite far as a device to understand operations of rules, and to make empirical predictions of measures observable on real descent sequences (that is, on real cultural systems), acting under rules.

We define an ordered list counting the numbers of regular structures present in a particular concrete configuration $C_t=(\mathcal{B}^t,\mathcal{M}^t)$ as a configuration vector or simply a configuration;

$$C := (m_0, m_1, m_2, \ldots, m_j, \ldots)$$

where the coefficient m_j is the number of elements of type Mj in C_t. Thus a configuration consisting only of 2 of the M2 structures would be written (0,0,2,0, …). Sometimes, for narrative purposes, to more explicitly specify a configuration that is only, for example, 2 of the M2 structures, we write the vector as (2(M2)) or simply 2(M2).

For finite generation size $\#\mathfrak{G}$, which is certainly the case for all known practical applications to cultures or living species, there will be some date t and some integer $\max(n) \le ½\#\mathfrak{G}^t$, above which size a regular structure Mn is not possible. We can thus create an ordered list

$$C=(m_0, m_1, m_2,\ldots, m_{\max(n)}).$$

whose values are simply the number of copies of each Mn in the picture of C_t. The entries 0 for $i>\max(n)$ are not shown but if needed we can always append additional 0’s after the $m_{\max(n)}$ entry. If C=(0,0,…,0) or (0,0,…,0,…) then on the underlying set $\mathfrak{G}^t$ there are no regular structures of any length. Since we are considering only regular structures, such generation is also therefore empty. When we write C=0 we mean, the vector C=(0,0,0,…,0). Of course if C=0 and thus $\mathfrak{G}^t=\varnothing$, this means also, the sequence becomes extinct.[17]

[16] We do not depict an M0 structure, which would be a single dot enclosed within a single circle. This might represent for example, a “self-reproducing” organism, such as a single cell. An M1 structure would be two dots connected by a line, all enclosed within a circle. Thus would represent a structure that allowed “incest” among members of a sibship, for example. We include the M0 and M1 structures in the vector for theoretical completeness, but in other papers will discuss their implications. Most human cultures prohibit marriages of a brother-sister pair, thus require at least an M2 structure to represent their operation.

[17] One question not treated above is, what is the configuration on a non-empty generation $\mathfrak{G}^t$ of a descent sequence $\mathcal{S}$ of some evolutionary structure $\mathbf{S}$, when the set $\mathbb{W}$ of all relations defined on the evolutionary structure is empty but $\mathfrak{G}^t$ is not? This condition is impossible. If $\mathbb{W}$ is empty then there is no *B* relationship, thus no individuals who have descended from any ancestors, and thus $\mathfrak{G}^t$ is necessarily empty, contradicting the requirement that $\mathfrak{G}^t$ is not empty. Thus if $\mathfrak{G}^t$ is a generation and it is not empty, then necessarily it is accompanied by a nonempty $\mathbb{W}$ that contains at least *B*.

Also, since in counting configurations we consider only M-sets that have offspring, then there are as many B-sets in the immediate decedent generation of any particular generation, as there are M-sets in the configuration of the parental generation. We leave for later papers to consider the effects of relaxing these restrictions.

Let $\mathbb{C}$:={C | C is a configuration vector} denote a set of configuration vectors.

7.2 Addition of Generations and of Configurations

Definition 6: Let $\mathbb{S}$=($\mathbb{P}$, $\mathbb{R}$, *D, B, M*) be an evolutionary structure with common calendar $t_s \in \mathcal{T}_s$, and let $\mathcal{G} = \{\mathbb{G}^i \mid i \in \mathcal{T}_G\}$ and $\mathcal{H} = \{\mathbb{H}^j \mid j \in \mathcal{T}_H\}$, be independent descent sequences of $\mathbb{S}$ with local calendars $\mathcal{T}_G$, and $\mathcal{T}_H$ respectively, having calendar constants γ and η respectively, $\gamma=\eta$, local dates $i \in \mathcal{T}_G$ and $j \in \mathcal{T}_H$. If $i+\gamma=j+\eta=t \in \mathcal{T}_s$, then "+' is defined, as:

$$\mathbb{G}^t+\mathbb{H}^t := \mathbb{G}^t \cup \mathbb{H}^t = \mathbb{K}^t,$$

Then stating dates in the common calendar, since

$$\{\mathbb{G}^t \cup \mathbb{H}^t, \mathbb{G}^{t+1} \cup \mathbb{H}^{t+1}, \ldots\} = \{\mathbb{K}^t, \mathbb{K}^{t+1}, \ldots\}$$

if we let $\mathcal{K}=\{\mathbb{K}^t, \mathbb{K}^{t+1}, \ldots\}$ so that

$$\mathcal{K}=\mathcal{G}+\mathcal{H} := \{\mathbb{G}^t+\mathbb{H}^t, \mathbb{G}^{t-1}+\mathbb{H}^{t+1}, \ldots\}$$

then $\mathcal{K}$ is a descent sequence with calendar constant τ=t. The definition requires $\gamma=\eta$ since both descent sequences start with their local calendar date 0, and we wish to add only generations with the same common calendar date. Since we require that $\mathcal{G}$ and $\mathcal{H}$ are independent sequences, then the respective populations are such that $\mathbb{G} \cap \mathbb{H} = \varnothing$ and therefore $\mathcal{G} \cap \mathcal{H} = \varnothing$ so also for each t in their common calendar, $\mathbb{G}^t \cap \mathbb{H}^t = \varnothing$. Therefore, if $\mathbb{G}^t$ has configuration C and $\mathbb{H}^t$ has configuration D then the sum $\mathbb{G}^t+\mathbb{H}^t=\mathbb{K}^t$ has configuration E, C+D=E where "+" is ordinary vector addition.

Given any two configurations C and D, we may therefore think of the vector sum C+D of the configurations C and D as the configuration corresponding to $\mathbb{G}^t+\mathbb{H}^t$ obtained when any two independent descent sequences $\mathcal{G}$ and $\mathcal{H}$ present C for $\mathbb{G}^t$ and D for $\mathbb{H}^t$. This allows us to interpret the vector sum C+D of configurations in terms of sums of descent sequences without knowing the sequences in advance (so long as the two generations share a common calendar date). Note that C+C then represents the outcome of taking two descent sequences, each manifesting C in their respective common calendar generation t, where the populations at date t are disjoint and there is no identification of rules of one descent sequence with corresponding rules in the other descent sequence.

Thus let $\mathbb{C}$ be a set of configurations and let C,D,E $\in \mathbb{C}$. If C=(c_0, c_1,…, c_j, …) and D=(d_0, d_1, …, d_j, …). Then

$$C+D := (c_0+d_0, c_1+d_1, \ldots, c_j+d_j \ldots)$$

So, if C is a 2(M2) configuration and D is a 3(M2) configuration then the sum has 5(M2), or in vector form:

$$C+D = (0,0,2,0,0,\ldots)+(0,0,3,0,0,\ldots)=(0,0,5,0,0,\ldots).$$

Thus the operation "+" on configurations is commutative (since ordinary arithmetic addition of non-negative integers[18] commutes, so C+D=D+C); associative (since ordinary arithmetic addition of the non-negative integers is associative so [C+D]+E=C+[D+E]); and an additive identity 0 exists (since $0 \in \mathbb{C}$ and if $C \in \mathbb{C}$ then C+0=0+C=C). If C=D, then of course C+D=C+C=2C. So also, multiplication of C by an integer n simply adds n copies of C, thus by ordinary addition: $nm_j=m_j+m_j+\ldots$, performed n times for each element of C; we notate the result by nC. From this it follows that if $C,D \in \mathbb{C}$ and n is a non-negative integer, then $nC \in \mathbb{C}$, and also $n(C+D)=(nC+nD)=nC+nD=E \in \mathbb{C}$, is just addition of two vectors and is also in $\mathbb{C}$.

8 Rules and Rule Transforms

We now study the results of applying a rule to a generation in a descent sequence, on the possible configurations in the next generation. Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $R \in \mathbb{R}$ be a rule, let $\mathcal{G}$ be a descent sequence of $\mathbb{S}$, let $\mathbb{G}^t \in \mathcal{G}$, let $\mathbb{C}$ be a set of configurations on generations in $\mathbb{S}$, and let $C \in \mathbb{C}$ be the configuration on $\mathbb{G}^t \in \mathcal{G}$. Then RC will mean an effect of the rule R acting on C in creating some configuration, say $D \in \mathbb{C}$, on generation $\mathbb{G}^{t+1}$. Since we write $\mathbf{R}C=\{D \mid D \in \mathbb{C}$, D is allowed by application of rule R to some $C \in \mathbb{C}\}$ to mean a particular set of allowed result(s) of application of rule R to C in one step, we can write R^2C to mean the result of application of R to a sequence starting with C for two successive generations. That is $R^2C=RRC$, where $RRC=\{E \mid$ for some $D,E \in \mathbb{C}$, $E \in RD$, $D \in RC\}$; thus RRC is the set of configurations allowed following two successive applications of rule R starting from configuration C. And generically write R^kC for the application of R for k successive generations starting from C.

We formalize these notions, including the possibility that RC may have more than one allowed outcome:

Definition 7:[19] Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, let $\mathbb{C}$ be the set of configurations of $\mathbb{S}$, let $C,D \in \mathbb{C}$, and let $R \in \mathbb{R}$. Then

1. (i) $r_{DC}:=1$ iff R allows a transition from C to D, and
   (ii) $r_{DC}:=0$ iff R does not allow a transition from C to D.
2. In application of Definition 7.1 establish an arbitrary standard order for listing the $C \in \mathbb{C}$. Then $\mathbf{R}=[r_{DC}]$ is a square array on the ordered pairs (D,C) $C \in \mathbb{C}$, $D \in \mathbb{C}$ called the <u>transform</u> of R.

---

[18] Because we are for now dealing only with regular structures, and finite generations, all coefficients are non-negative integers.

[19] See Definitions 13 and 14 of Ballonoff, (2008), footnote 8. Note that the operation of transforms depends on what the rules allow. Except that we have defined the relations *M*, *B* and *D*, we have not explicitly stated what it is that rules can or can not allow. Implicitly, we assume that rules are stated in terms of the configurations on which they are currently acting. We do not prohibit that their decisions may depend also on configurations in previous generations, and on other facts. We leave for later papers to develop a more complete theory of statement and expression of rules and their transforms.

3. Let:

$$\mathbf{R}C := \{D \mid D \in \mathbb{C} \text{ and } r_{DC}=1\},\ \mathbf{R} \text{ the transform for } R,\ C \in \mathbb{C}.$$

Then **R**C is the <u>set of accessible configurations</u> under R starting from C.

4. Let $\check{\mathbf{R}} := \{\mathbf{R} \mid R \in \mathbb{R}, \mathbf{R}$ is a rule operator for R$\}$ be the set of transforms of $\mathbb{R}$.
5. Let $R,S \in \mathbb{R}$ and let $\mathbf{R},\mathbf{S} \in \check{\mathbf{R}}$ be the corresponding transforms. Then:

$$\mathbf{RS}C := \bigcup_{D \in \mathbf{S}C} \mathbf{R}D.$$

Note that any such **R** is thus a map $\mathbf{R}: \mathbb{C} \rightarrow \mathbb{C}$. Following Definition 7.4 if we pick R twice, then $\mathbf{RR}C = \bigcup_{D \in \mathbf{R}C} \mathbf{R}D$. We use $\mathbf{R}^k C$ to denote the set which is the result of all the possible chains of outcome of k sequential applications of R starting from C. If we write **R**C=D we imply that the set **R**C={D} has only that member D. Write $RC \Rightarrow D$ if the result of applying R has selected the specific result D (in this particular instance), which also requires that $D \in \mathbf{R}C$, but does not require that **R**C=D (though certainly is consistent with that condition).

9 Minimal Structures

Let R be a set of rules, let $R \in R$ and let R be the transform for R. Let k be an integer, k>0, and let $C \in \mathbf{R}^k C$. If the number of generations k for the first such allowed reproduction of a configuration in the shortest such sequence is k=n generations, then R is said to be n-stable. Often, that is in one generation, in which case R is 1-stable. If C is a configuration, the size of C is simply the population size #C of the generation on which C is described.

> *Definition 19*: Let $\mathbb{S}=(\mathbb{P}, \mathbb{R}, D, B, M)$ be an evolutionary structure, Let $R \in \mathbb{R}$ be a rule, let $\mathbb{C}$ be the set of configurations on the generations of descent sequences in $\mathbb{S}$ let **R** be the transform for R. Let
>
> $$\mathbb{C}_R := \{C \mid C \in \mathbf{R}^k C,\ k>0,\ C \text{ is k-stable},\ C \in \mathbb{C}\}.$$
>
> Then $C \in \mathbb{C}_R$ is a <u>minimal structure of R</u> iff for all $D \in \mathbb{C}_R$, if **R**C is i-stable and **R**D is j-stable, both $i \leq j$ and $\gamma_C \leq \gamma_D$.

Simply stated, a minimal structure of a rule R, if one exists, is a smallest non-empty configuration which "reproduces itself" in the shortest number of generations under action of R. $\mathbb{C}_R$ is the set of configurations in $\mathbb{C}$ that can "reproduce themselves" under R. We denote a configuration C that is a minimal structure of R as M_R. Note that a minimal structure for a rule R exists provided $\mathbb{C}_R$ is non-empty.

All minimal structures (known to this author) for any marriage rule that prohibits marriage within the sibship, are composed of regular structures.[20] To the knowledge of this author as well, if an empirically known marriage rule has a 1-stable minimal structure, such structure is unique. In recognition of this empirical fact, we sometimes use the definite article "the" minimal structure

[20] See for example the diagrams used in Gould (2000) to represent kinship terminologies, mapping kin terms onto M2, 2(M2) or 4(M2). See footnote 15 for more explanation.

of a rule R. But it is easy to construct a (theoretical) counter example: Let R be the rule "each generation must have 2 marriages". Then there are two 1-stable minimal structures for this rule, 2(M1) and M2. If we however add an "incest" restriction, prohibiting marriage within a sibship (within a B-set), then examples with k(M1) k>0 disappear. It is an interesting empirical and thus theoretical question whether there are empirically occurring rules that have more than one minimal structure (apparently, the answer is "no"), and if not, why not.

There is however, a property of a minimal structure which is characteristic of the rule, and shared with all 1-stable minimal structures for that rule. Let $M_R= (m_0,m_1,\ldots,m_j, \ldots)$ be a minimal structure of a rule R. Let $s:=\Sigma(jm_j)$, which is thus simply the number of M-sets in the smallest size 1-stable configuration. We call such s the structural number of R. The structural number is an important property of a rule, since in connection with the density function of the Stirling Number of the Second Kind, it allows computation of predictions of empirically testable population statistics associated with the existence of that rule, and associated with transitions between rules.[21]

10 Fixed Point of a Transform

We are interested especially in rules for which there exists at least one configuration C for which $\mathbf{R}C \in \mathbf{RR}C$. A (non-trivial) fixed point of a transform $\mathbf{R}:\mathbb{C} \to \mathbb{C}$ on a non-empty set $\mathbb{C}$ of configurations is a configuration $C \neq 0$, $C \in \mathbb{C}$, such that $C \in \mathbf{R}C$.

> *Definition 10:* Let $\mathbb{C}$ be a non-empty set of configurations let $C \in \mathbb{C}$ and $C \neq 0$, let $R \in \mathbb{R}$ be a rule, $\mathbf{R} \in \check{\mathbf{R}}$, the rule transform corresponding to R, and let $kC \in \mathbf{R}(kC)$ where k is a positive integer. Then kC is a fixed point of $\mathbf{R}$.

It should be obvious that a fixed point of a transform is a 1-stable structure of the corresponding rule.[22] And, if a rule R has one or more fixed point, the rule is 1-stable, and a minimal structure is among those fixed points.

11 Comments About Rules and Configurations

One of our objectives is to find sufficiently compact representations for cultural objects that we can analyze their properties and predict properties of cultures, in some more generic sense than simply describing what John and Mary are doing today. The use of an abstract concept of

[21] See P. Ballonoff (1982a) "Mathematical Demography of Social Systems," *Progress in Cybernetics and Systems Research*, Vol. 10, Hemisphere Publishing, pp.101-112; and P. Ballonoff (1982b) "Mathematical Demography of Social Systems, II," in Trappl (ed.), *Cybernetics and Systems Research*, North-Holland, pp. 555-560.; and Ballonoff, P. 1974, "Statistical Theory of Marriage Structures" pages 11 - 27 in P. Ballonoff (ed) *Mathematical Models of Social and Cognitive Structures*, University of Illinois Press, Urbana. These papers can also be found on the "publications" page at http://www.Ballonoff.net.

[22] The existence of fixed points is a small sample of what it is possible to discover by this approach.

configurations is one device which allows this. When discussing the effects of rules on configurations, the discussion shifts from analysis of particular empirical subpopulations $\mathbb{G}^t$ to the sets of possible configurations $\mathbf{R}C \subseteq \mathbb{C}$. We have moved from considering concrete relationships among specific individuals, to considering the abstract sets of possible relations created by the operation of the rule.

It has long been the practice of ethnographers to draw 1-stable minimal structures to illustrate the operation of a marriage rule and/or the application of a kinship terminology of a culture.[23] Evidently, anthropology has long used 1-stable minimal structures without realizing the tremendous analytical power they imply. For example, it is sometimes claimed in anthropology that "rules" are not "real" because they can not be directly observed. But, their effects can be directly observed and measured. When ethnographers use minimal structures of rules to illustrate a kinship terminology, they are also offering a form of empirical verification (observation) of the existence of the rule. We can also test for the presence of a rule by observing the empirical values of "demographic" measures on generations, predicted from the structural number of the rule.[24]

12 Summary and Conclusion

We have laid out a theory of cultural rules as mathematical transforms (set functions). We have shown how rules may be represented as transforms, acting on ordered lists we call "vectors" and

[23] Again, see Gould (2000) for examples. But a great many ethnographies use similar diagrams.

[24] Following more detailed development of formulas in the papers cited in footnote 21. The original developments are based on two key arguments. Ballonoff (1974) assumed a binomial distribution of sex, on a generation of N individuals, with approximately a 50-50 sex ratio, in a system not restricted to be only made up of our configurational elements, and finds a relationship between the structural number of the rule, and the least "practical" number of "reproducing family lines" L. If we require one female ascribed as reproducing per each reproducing "line", then F=L. The paper then derived F in a generation of size N, asymptotic to the values $L \geq s$ as: $F=\ln\{(N-s)/N\}/\ln\{(N-1)/N\}$. It then finds a Stirling Number of the Second Kind density which has that value of F=L at the maximum density for distributing a population of size N among L non-empty cells, and thus finds a pair of values N and L given s. Dividing gives an average family size characteristic of s as $n_s=N/L$. Since half of N is female, then taking $p_s=L/(\frac{1}{2}N)=2L/N=1/(\frac{1}{2}n_s)$ gives a proportion of the female population that are ascribed as reproducing, which is characteristic of s. The above thus gives a pair (n_s,p_s) that is characteristic of s such that $n_sp_s=2$ or $\frac{1}{2}n_sp_s=1$.

Ballonoff (1982a, and 1982b) then develop the consequences of the fact that $1=e^{rT}$ when r=0, which allows us to write $\frac{1}{2}n_sp_s= e^{rT}$. Because this value of r can be considered a growth rate (or decline, depending on sign) the papers then show that the above logic allows to compute predictions of growth rates r associated with cultural change, when measuring n and p as linear combinations of the (n_s,p_s) values of rules with different structural numbers in effect at the specified generation. This interpretation makes empirically valid predictions of statistics associated with ethnographically reported marriage rules, as shown in Ballonoff (1982a, 1982b) and elsewhere. The papers then discuss that if all rules in use in a generation having more than one rule, have the same structural number, then r=0, otherwise, $r \neq 0$; and thus the theory predicts a computable amount of growth or decline due to cultural change, given the proportionate mix of structural numbers of rules used at a given generation. There are many sources of stochastic variance in this analysis which need to be isolated, including that caused by the shape of the SNSK density function, that caused by variation in sex ratio, and that resulting from the relatively discrete nature of the SNSK distribution at small values. For further discussion refer to the original papers.

having fixed points. A special smallest fixed point of a rule, the minimal structure of such rule, is certainly something that can be observed, and indeed anthropologists and ethnographers spend a great deal of effort observing descent maps **D** and describing their actions as "marriage rules", using exactly what we called the minimal structure, as the diagrammatic framework on which they describe the rule and the associated kinship terminology. Marriage rules have been a subject of mathematical study in social anthropology, since they lend themselves more easily to "formal" and mathematical techniques. But also we hardly find any cultural systems without a marriage rule. They deal intimately (in all senses) with the survival of the culture and of the descent sequence.[25]

When we compute a minimal structure of a rule, we are computing a property of the rule, we are not necessarily computing a prediction of the empirical network on an empirical population. In fact, unless the rule only allows a minimal structure configuration, it may be that a minimal structure of a rule seldom or even never forms as an empirical configuration in a descent sequence following that rule. Nonetheless properties of the minimal structure, in particular its structural number, predict other measurable observations on a descent sequence using a rule with that minimal structure (as summarized in footnote 25). Yet despite the long use of minimal structures for wholly descriptive purposes in ethnographies, social anthropology has never previously drawn inferences such as those just summarized. This emphasizes why to obtain a predictive theory, we must do much more than the purely descriptive approach of traditional ethnography.

Paul Halmos, in his classic work *Introduction to Hilbert Space*[26], justifies simplifying the apparent complexity of Hermitian operators by looking at their real projections, this way: "The purpose of such an operational approximation theorem is … to provide a tool for deriving and understanding the deep structural properties of complicated objects in terms of simple objects." The motivation here is similar. We wish to demonstrate a theory that can handle the complexity of cultural systems. In doing so, we also use the occasional reduction of the example to very simple ones, to aid understanding, but equally importantly, to draw strong inferences. The fact that an apparently complex topic lends itself to reduction to intuitively simple cases (such as the minimal structures of rules and the properties derivable from such structures), which have predictive value in the correct way, lends credence to the belief that the overall theory being constructed is a correct one. We have hardly exploited even a small fraction of the rich potential opened by this approach.

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[25] A good example again comes from the Australian systems. Gould (2000) at page 4 cites George Murdock (1949) *Social Structure*, Macmillan, New York, at page 46, to the effect that an Australian aboriginal could traverse the entire continent, and by asking just a few questions on kinship terminologies, determine exactly how that person relates in many different ways to the residents of remote locations from the individual's place of origin. Such systems may survive longer in real time, or be viable over more successive generations, or apply over larger expanses of potentially interacting sub-populations, because of how the rule acts in regard to addition of very small populations.

[26] At page 57, Paul Halmos, *Introduction to Hilbert Space, and the Theory of Spectral Multiplicity*, 2nd edition, 1957 AMS Chelsea Publishing, Providence RI.

Table of Symbols Used

SYMBOL	PAGE FIRST USED	NAME/MEANING
$\mathbf{P}_i$	4	a non-empty set whose members are called <u>individuals</u>.
$\mathbf{S}_i$	5	An <u>evolutionary structure</u> $\mathbf{S}_i$ is a quintuple ($\mathbf{P}_i$, $\mathbf{R}_i$, D_i, B_i, M_i) where $\mathbf{R}_i$ is a non-empty set of rules, and D_i, B_i, and M_i are binary relations on $\mathbf{P}_i$.
$\mathbf{R}_i$,	5	is a non-empty set of <u>rules</u>. A rule is a statement of how a D_i , B_i, or M_i relationship may form that does not violate the conditions of Definition 1..
D_i	5	A totally non-symmetric and transitive binary relation on $\mathbf{P}_i$, called <u>descent</u>.
B_i	5	A binary relation on $\mathbf{P}_i$, such that if $b,c,d \in \mathbf{P}_i$, and both dP_ib and dP_ic then bB_ic, called <u>sibling of</u>.
M_i	5	A transitive and symmetric binary relation on $\mathbf{P}_i$, such that $\#bM_i \leq 2$, called <u>marriage</u>.
P	5	A binary relation on $\mathbf{P}_i$, such that if bD_ic and there exists no $d \in \mathbf{P}_i$, $d \neq b,c$ for which bD_id and dD_ic, then c is a <u>parent</u> of b and b is an <u>offspring</u> of c
$\mathbf{G}^t$, $\mathbf{H}^t$, $\mathbf{L}^t$,	6	A non-empty subset of $\mathbf{P}$, called a generation,
$\mathcal{T}$	6	A non-empty set $t \in \mathcal{T}$ of consecutive non-negative integers whose first inter is t=0, called the <u>local calendar</u> of a descent sequence.
$\mathcal{G}$	6	$\mathcal{G}=\{\mathbf{G}^t \mid t \in \mathcal{T}\}$ is called a <u>descent sequence</u> of $\mathbf{S}$ in case, for all $\mathbf{G}^t \in \mathcal{G}$, M and B respect $\mathbf{G}^t$; the pairs D,B and D,M split $\mathbf{G}^t$; and when $\mathbf{G}^{t-1},\mathbf{G}^t \in \mathcal{G}$; $b \in \mathbf{G}^t$; and cPb, then $c \in \mathbf{G}^{t-}$
$\mathcal{G}_{ij}$	6	the j[th] descent sequence of $\mathbf{S}_i$
$\mathbf{G}_{ij}{}^t$	6	the t[th] generation of $\mathcal{G}_{ij}$
$\mathcal{T}_s$	6	A non-empty set $t_S \in \mathcal{T}_S$ of consecutive non-negative integers whose first inter is t_S=0, called the <u>common calendar</u> of a descent sequence $\mathbf{S}$.
t	6	$t \in \mathcal{T}$ is a <u>date</u> in the local calendar $\mathcal{T}$. The date of the generation denoted as ${}_\kappa\mathbf{G}_{ij}{}^0$ in the common calendar.
t_S	6	$t_S \in \mathcal{T}_S$ is a <u>date</u> in the common calendar $\mathcal{T}_S$.
κ (sometimes also γ, η, λ)	7	A non-negative integer associated with a descent sequence $\mathcal{G}$ called the <u>calendar constant</u> of $\mathcal{G}$.
${}_\kappa\mathcal{G}_{ij}$	7	The j[th] descent sequence of an evolutionary structure $\mathbf{S}_i$, having calendar constant κ.
${}_\kappa\mathbf{G}_{ij}{}^t$	7	${}_\kappa\mathbf{G}_{ij}{}^t \in {}_\kappa\mathcal{G}_{ij}$ The t[th] generation of a descent sequence ${}_\kappa\mathcal{G}_{ij}$.
${}_\kappa\mathbf{G}_{ij}{}^0$	7	${}_\kappa\mathbf{G}_{ij}{}^0 \in {}_\kappa\mathcal{G}_{ij}$ The initial generation of a descent sequence ${}_\kappa\mathcal{G}_{ij}$.

$\mathfrak{M}^*$	7	$\mathfrak{M}^* := \{bM \mid b \in \mathbf{P}\}$ be the set of all marriages in the evolutionary structure **S**
$\mathfrak{M}$	7	$\mathfrak{M} := \{M \mid M \in \mathfrak{M}^*$, and for $b \in M\ \exists(d)(d \in \mathbf{P}$ and $dDb)\}$, the set of all reproducing marriages in **S**
$\mathfrak{B}$	7	$\mathfrak{B} := \{bB \mid b \in \mathbf{P}\}$ be the set of all sibships in the evolutionary structure **S**
$\mathfrak{M}^t$	7	$\mathfrak{M}^t = \{M \mid M \in \mathfrak{M}$ and $M \subseteq \mathbf{G}^t\}$ be the set of all marriages in the t[th] generation of the descent sequence $\mathcal{G}$ of **S**,
$\mathfrak{B}^t$	7	$\mathfrak{B}^t := \{B \mid B \in \mathfrak{B}$ and $B \subseteq \mathbf{G}^t\}$ be the set of all sibships in the t[th] generation of the descent sequence $\mathcal{G}$ of **S**
$\mathbf{G}$	7	$\mathbf{G} := \{b \mid b \in \mathbf{G}^t, t \in \mathcal{T}\}$ be the <u>population of the descent sequence</u> $\mathcal{G}$ of the evolutionary structure **S**.
$\mathcal{T}_G$	8	The local calendar of descent sequence $\mathcal{G}$.
$\mathcal{T}_H$	8	The local calendar of descent sequence $\mathcal{H}$.
$\mathcal{T}_K$	8	The local calendar of descent sequence $\mathcal{K}$.
$\mathcal{G}_c$	9	For individual $c \in \mathbf{P}$, of evolutionary structure **S**, a descent sequence of **S** such that $\mathbf{G}_c{}^t \in \mathcal{G}_c$.
$_\kappa\mathcal{G}_c$	9	A descent sequence $\mathcal{G}_c$ containing individual c, having calendar constant κ.
$\mathcal{C}$,	9	$\mathcal{C} = {_\kappa\mathcal{G}_c}$, a name for the descent sequence $_\kappa\mathcal{G}_c$
$\mathcal{D}$	9	$\mathcal{D} = {_\kappa\mathcal{G}_d}$, a name for the descent sequence $_\kappa\mathcal{G}_d$
γ^t	11	$\gamma^t := \#\mathbf{G}^t$
β^t	11	$\beta^t := \#\mathfrak{B}^t$
μ^t	11	$\mu^t := \#\mathfrak{M}^t$
T	12	A non-negative real number called a <u>generation interval</u>.
C_t	12	$C_t := (\mathfrak{B}^t, \mathfrak{M}^t)$ is the pair consisting of the partition $\mathfrak{B}^t$ and the sets $M \in \mathfrak{M}^t$ on a generation $\mathbf{G}^t$ of a descent sequence $\mathcal{G}$ of an evolutionary structure **S**, called a <u>concrete configuration.</u>
Mn, M1, M2, etc.	13	Mn is a name for a configuration (a "regular structure") consisting of a loop of alternating M and B elations, containing n of the M relations.
m_j	14	The number of elements of type Mj in configuration C
max(n)	14	$\max(n) \leq \frac{1}{2}\#\mathbf{G}^t$, the maximum n of an object Mn in a generation of size $\#\mathbf{G}^t$
C	14	$C = (m_0, m_1, \ldots, m_{\max(n)})$, an ordered list of the number of each regular structure Mn in a particular concrete configuration.
RC	16	Notation for the application of rule R to a generation described by configuration C
RRC	16	Notation for the application of rule R for two successive

		generations to a generation described by configuration C
R^kC	16	Notation for the application of rule R for k successive generations to a generation described by configuration C
r_{DC}	16	$r_{DC}=1$ if rule R allows a transition from configuration C to configuration D, $r_{DC}=0$ if rule R does not allow a transition from configuration C to configuration D
R	16	$\mathbf{R}:=[r_{DC}]$ a square array of the values r_{DC} on the ordered pairs (D,C), $D,C \in \mathbb{C}$, defined by a rule R, called the <u>transform</u> of R.
$\check{\mathbf{R}}$	17	$\check{\mathbf{R}} := \{\mathbf{R} \mid \mathbb{R}$ a set of rules, $R \in \mathbb{R}$, and $\mathbf{R}$ is a transform for R$\}$
RC	17	Given a rule R, a set of configurations $\mathbb{C}$, and a configuration $C \in \mathbb{C}$, the set $\{D \mid D \in \mathbb{C}$ and $R_{C,D}=1\}$ is the set of configurations of a descent sequence $\mathcal{S}$, that may follow in one generation from configuration C under rule R..
RRC	17	Given a rule R, $\{E \mid D,E \in \mathbb{C}, D \in \mathbf{R}C, E \in \mathbf{R}D$, for some $C \in \mathbb{C}. \}$. Showing the effect of applying R for two successive generations starting from C.
$\mathbf{R}^kC$	17	Given a rule R, a set of configurations $\mathbb{C}$, and a configuration $C \in \mathbb{C}$, the set $\{E \mid C,D,E .. \in \mathbb{C}, D \in \mathbf{R}C, E \in \mathbf{R}D. \ldots$ as allowed by rule R applied for k successive generations$\}$.
$\Rightarrow$	17	$RC \Rightarrow D$ is a particular $D \in \mathbf{R}C$ which occurs on a generation at date t+1 from among the set of those allowed following application of R to a generation at date t having configuration C.
γ_C	17	the <u>size</u> of C, the size of the underlying generation on which C is defined.
$\mathbb{C}_R$	17	$\mathbb{C}_R := \{C \mid C \in \mathbf{R}^kC, k>0$, C is k-stable, $C \in \mathbb{C}\}$ the set of configurations in a set of configurations $C \in \mathbb{C}$ that can recur after k>0 applications in sequence of a rule R.
M_R	17	A configuration $C \in \mathbb{C}_R$ such that for all $D \in \mathbb{C}_R$, when $C \in \mathbf{R}^iC \in \mathbb{C}_R$ and $D \in \mathbf{R}^jD \in \mathbb{C}_R$, both $i \leq j$ and $\nu_C \leq \nu_D$, called a minimal structure of R.
s	18	If $M_R = (a_0, a_1, \ldots, a_j, \ldots)$ is a minimal structure of a rule R then $s := \Sigma(ja_j)$ is the structural number of R.